Q3 EUF 2014 2sem

$$\rho(\nu)\,d\nu = \frac{8\pi\nu^2}{c^3}\cdot\frac{d\nu}{e^{h\nu/kT}-1}$$ densidade de energia

a) $\lambda = \frac{c}{\nu} \Rightarrow \nu = \frac{c}{\lambda}$ $\quad B_\nu(T) = \frac{2h\nu^3}{c^2}\,\frac{1}{e^{h\nu/kT}-1}$

$$d\lambda = -\frac{c\,d\nu}{\nu^2} \Rightarrow d\nu = -\frac{d\lambda\,\nu^2}{c} = -\frac{d\lambda\, c}{\lambda^2} \quad B(T) = B_\nu\, d\nu = \left(\frac{c}{\lambda}\right)^3\cdot\left(-\frac{d\lambda\, c}{\lambda^2}\right)$$

$$\rho(\lambda)\,d\lambda = -\frac{8\pi}{\lambda^4}\,\frac{d\lambda}{\left(e^{\frac{hc}{\lambda k_BT}}-1\right)}$$

b) $\lambda \to \infty, T \to \infty$ ($\gg 1$) $\quad \rho(\lambda)\,d\lambda = \frac{-8\pi\, d\lambda\, k_BT}{hc\lambda^4 + hc}$

$$e^{\frac{hc}{\lambda k_BT}} \approx 1 + \frac{hc}{\lambda k_BT}$$

$$\boxed{\rho(\lambda)\,d\lambda = \frac{-8\pi\, d\lambda\, k_BT}{4\lambda^3\cdot hc}}$$

c) Stefan-Boltzmann law $\Rightarrow j^* = \sigma T^4$ densidade

$R_T = \frac{P}{A} = \int R(\lambda)\,d\lambda \int d\Omega$ ângulo sólido $\quad R_T = \sigma T^4 \Rightarrow$ power radiated from a black body

$\boxed{P = A R_T}$ $\quad R(\lambda) = c\cdot\frac{\rho(\lambda)}{4}$

$$= \int_0^\infty \frac{-8\pi c}{\lambda^4}\,\frac{d\lambda}{\left(e^{\frac{hc}{\lambda k_BT}}-1\right)} = \; = /$$

Q4.

A: $v_0 \rightarrow$ $\quad -v_0 \leftarrow$

B: Rest $\quad -v' \leftarrow$

Inertial

energia cinética

$$K = E - mc^2$$

energia total $\quad$ energia repouso

$\hookrightarrow \gamma m_0 c^2$

a) $$v_x' = \frac{v_x - V}{1 - \frac{V v_x}{c^2}} = \frac{-v_x - v}{1 + \frac{v^2}{c^2}} = \boxed{\frac{-2v}{1 + \frac{v^2}{c^2}}}$$ ✓

eles param depois da colisão

b) B Antes

$$p_i = \gamma m \cdot v_x' = -m \frac{2v}{1 + \frac{v^2}{c^2}} \cdot \frac{1}{\sqrt{1 - \frac{v_x'^2}{c^2}}} = -m \frac{v_x'}{\sqrt{1 - \frac{v_x'^2}{c^2}}}$$

$v_A = 0$

Após,

$v_B = v$

$v_A = 0$

$$p_f = \gamma_2 M v = \frac{2 M v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

c) conservação momento $\Rightarrow$ $$-\frac{m 2v}{\cancel{1 + \frac{v^2}{c^2}}} = Mv \Rightarrow M = -\frac{m2}{\cancel{1 - \frac{v^2}{c^2}}}$$

vezes 2 pq 2 partículas com velocidade iguais

A

$$E_i = \gamma m_0 c^2 = 2 \cdot \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

mov. $\quad$ parada

$$\Rightarrow E_1^B = \gamma_1 m_0 c^2 + m_0 c^2 = \left(\frac{m_0}{\sqrt{1 - \frac{v_x'^2}{c^2}}} + m_0\right) c^2$$

B

$$E_f = 2\gamma_2 M c^2 = \frac{2 M c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$E_i = E_f$

$$\gamma_1 m_0 \cancel{c^2} + m_0 \cancel{c^2} = 2\gamma_2 M \cancel{c^2}$$

$$\boxed{M = \frac{m_0}{2\gamma_2}(\gamma_1 + 1)}$$

Q9.

$V = \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$

a)

$$-\frac{\hbar}{2m} \nabla^2 \psi + V\psi = E\psi$$

$$\psi' = -\alpha r C e^{-\frac{\alpha r^2}{2}}$$

$$\psi'' = -\alpha C e^{-\frac{\alpha r^2}{2}} + \alpha^2 r^2 C e^{-\frac{\alpha r^2}{2}}$$

suppose $\psi = C e^{-\alpha r^2/2}$

$$-\frac{\hbar}{2m}\left[-\alpha C \cancel{e^{-\frac{\alpha r^2}{2}}} + \alpha^2 r^2 C \cancel{e^{-\frac{\alpha r^2}{2}}}\right] + \left(\frac{1}{2} m\omega^2 r^2 - E\right) C \cancel{e^{-\frac{\alpha r^2}{2}}} = 0$$

$$-\alpha \cancel{C} + \alpha^2 r^2 \cancel{C} - \frac{m^2\omega^2 r^2}{\hbar} \cancel{C} + \frac{2mE}{\hbar} \cancel{C} = 0$$

1D

define

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \text{ and } \hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

$$\hat{H}\psi = E\psi \quad ; \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \Rightarrow \hat{H} = \hbar\omega\left[\frac{\hat{p}^2}{2m\hbar\omega} + \frac{m\omega \hat{x}^2}{2\hbar}\right]$$

$$\hat{a}\hat{a}^{+} = \frac{m\omega}{2\hbar}\left(\hat{x}^2 + \frac{\hat{p}^2}{(m\omega)^2} - \frac{i\hat{x}\hat{p}}{m\omega} + \frac{i\hat{p}\hat{x}}{m\omega}\right)$$

$$\hat{H} = \hbar\omega\left[\hat{a}^{+}\hat{a} + \frac{1}{2}\right]$$

$$= \frac{m\omega\hat{x}^2}{2\hbar} + \frac{\hat{p}^2}{2\hbar m\omega} + \frac{i}{2\hbar}[\hat{p},\hat{x}] = \frac{m\omega\hat{x}^2}{2\hbar} + \frac{\hat{p}^2}{2\hbar m\omega} + \frac{1}{2}$$

$[\hat{x},\hat{p}] = i\hbar$

$\hat H = \frac{\hbar\omega}{2}(2\hat a \hat a^\dagger + 1)$; $[\hat H, a^\dagger] = \left(\hbar\omega \hat a \hat a^\dagger + \frac{\hbar\omega}{2}\right) a^\dagger - a^\dagger \cdot \left(\hbar\omega \hat a \hat a^\dagger + \frac{\hbar\omega}{2}\right)$

$[\hat a, \hat a^\dagger] = +1$

$$= \hbar\omega(\hat a \hat a^\dagger \hat a^\dagger - \hat a^\dagger \hat a \hat a^\dagger) + \cancel{\frac{\hbar\omega}{2}}\hat a^\dagger - \hat a^\dagger \cancel{\frac{\hbar\omega}{2}}$$

$$= \hbar\omega(\hat a \hat a^\dagger - \hat a^\dagger \hat a)\hat a^\dagger = \hbar\omega[\hat a, \hat a^\dagger]\hat a^\dagger = +\hbar\omega \hat a^\dagger$$

$$-\hat a^\dagger \hat a = -\frac{m\omega}{2\hbar}\left(\hat x - \frac{i\hat p}{m\omega}\right)\left(\hat x + \frac{i\hat p}{m\omega}\right) = -\frac{m\omega}{2\hbar}\left[\hat x^2 + \frac{\hat p^2}{m\omega} - \frac{i\hat p\hat x}{m\omega} + \frac{i\hat x\hat p}{m\omega}\right] = -\frac{m\omega\hat x^2}{2\hbar} - \frac{\hat p^2}{2\hbar} - \frac{i}{2\hbar}\overset{+\frac{1}{2}}{[x,p]}$$

Grand state E_0

$$E_0 = \langle 0|\hat H|0\rangle = \langle 0|\frac{\hbar\omega}{2}(2\hat a^\dagger \hat a^\dagger + 1)|0\rangle = \langle 0|\hbar\omega \hat a^\dagger \hat a|0\rangle + \langle 0|\frac{\hbar\omega}{2}|0\rangle$$

$$= \langle 0|\hbar\omega \hat a^\dagger(\hat a|0\rangle) + \frac{\hbar\omega}{2}\langle 0|0\rangle = 0 + \frac{\hbar\omega}{2} = \boxed{\frac{\hbar\omega}{2}}$$

$\hat a|0\rangle = 0$

ground state

$$\hat H \hat a^\dagger = \hat a^\dagger \hat H - \underbrace{\hat a^\dagger \hat H + \hat H \hat a^\dagger}_{[\hat H, \hat a^\dagger]}$$

$$\hat H \hat a^\dagger|E_n\rangle = \hat a^\dagger \hat H|E_n\rangle + [\hat H, \hat a^\dagger]|E_n\rangle = \hat a^\dagger E_n|E_n\rangle + \hbar\omega \hat a^\dagger|E_n\rangle = (E_n + \hbar\omega)(a^\dagger|E_n\rangle)$$

energy increase $\hbar\omega$

$$\hat H(\hat a^\dagger(\hat a^\dagger|E_n\rangle)) = \underbrace{(E_n + \hbar\omega + \hbar\omega)}_{E_n + 2\hbar\omega}[a^\dagger(a^\dagger|E_n\rangle)]$$

starting with ground state: $\frac{\hbar\omega}{2}, \frac{3\hbar\omega}{2}, \frac{5\hbar\omega}{2}$

$$\hat H|0\rangle = \frac{\hbar\omega}{2}|0\rangle \Rightarrow \hat H a^\dagger|0\rangle = \left(\frac{\hbar\omega}{2} + \hbar\omega\right)|0\rangle$$

$$\hat H a^\dagger a^\dagger|0\rangle = \left(\frac{\hbar\omega}{2} + 2\hbar\omega\right)|0\rangle$$

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega}$$

Q9. a) $\boxed{\text{for 3D}}$ $H = \hbar\omega \sum_{i=1}^{3} \left(a_i^{\dagger} a_i + \frac{1}{2}\right)$

$$E_0 = \langle 0|\hat{H}|0\rangle = \hbar\omega\left[\langle 0|\sum_{i=1}^{3}\left(a_i^{\dagger}a_i + \frac{1}{2}\right)|0\rangle\right] = \hbar\omega\left[\langle 0|\sum_{i=1}^{3} \cancel{a_i^{\dagger}a_i}|0\rangle + \langle 0|\frac{3}{2}|0\rangle\right]$$

$$\boxed{E_0 = \frac{3\hbar\omega}{2}}$$

$$+\hat{H}\sum_{i=1}^{3} a_i^{\dagger} - \sum_{i=1}^{3} a_i^{\dagger}\hat{H} = \left[\hat{H}, (a_1^{\dagger} + a_2^{\dagger} + a_3^{\dagger})\right] = \hbar\omega \sum_{i=1}^{3} \hat{a}_i^{\dagger}$$

$$\therefore \Rightarrow \boxed{E_n = E_x + E_y + E_z = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega}$$

b) degenerescência: $n = n_1 + n_2 + n_3$ $\{ n_2 + n_3 = n - n_1$

n	E	degenerescência (3D)
0	$\frac{3}{2}\hbar\omega$	0
1	$\frac{5}{2}\hbar\omega$	3
2	$\frac{7}{2}\hbar\omega$	6

1, 0, 0
0, 1, 0
0, 0, 1

2, 0, 0
0, 2, 0
0, 0, 2
1, 1, 0
1, 0, 1
0, 1, 1

$\frac{1}{2}(n+1)(n+2)$

c) $\nabla^2\psi = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\psi\right]$ $\quad L^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) - \frac{L^2}{\hbar^2 r^2}\psi\right] + \frac{1}{2}m\omega^2 r^2 = E\psi$$

$$-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2}\Psi + \frac{1}{2}m\omega^2 r^2\,\Psi = E\Psi$$

$u = rR$ $\quad du = R + \frac{\partial R}{\partial r}$ $\quad d\dot{u} = \frac{\partial R}{\partial r} + \frac{\partial^2 r}{\partial r^2}$

Let $\boxed{\Psi_{nlm} = R_{nl}(r)\cdot Y_{l,m}(\theta,\phi)}$

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} = \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2 R}{\partial r^2}\right)$$

$$\cancel{Y_{lm}(\theta,\phi)}\cdot -\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{nl}(r)}{\partial r}\right) + \underbrace{\left[\frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2 l(l+1)}{2mr^2}\right]}_{V_{eff}}\cancel{Y_{lm}}\cdot R(r) = E\cancel{Y}R$$

$$\boxed{= -\frac{\hbar^2}{2mr^2}\left[2r\cdot\frac{dR(r)}{dr} + r^2\frac{d^2R(r)}{dr^2}\right] + \left(\frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2 l(l+1)}{2mr^2}\right)R(r) = E\,R(r)}$$

d) $l = 0$, $R(r) = e^{-\alpha r^2}$; $\frac{dR}{dr} = -2r\alpha e^{-\alpha r^2}$; $\frac{d^2R}{dr^2} = -2\alpha e + 4r^2\alpha^2 e^{-\alpha r^2}$

$$-\frac{\hbar^2}{m}\left[-2r\alpha \cancel{e^{-\alpha r^2}} + \left(-2r^2\alpha\cancel{e^{-\alpha r^2}} + 4r^4\alpha^2\cancel{e^{-\alpha r^2}}\right)\right] + \left[\frac{1}{2}m\omega^2 r^2\right]\cdot\cancel{e^{-\alpha r^2}} = E_n\cancel{e^{-\alpha r^2}}$$

$$E_n = \frac{2\hbar^2 r\alpha}{m} + \frac{2\hbar^2 r^2\alpha}{m} - \frac{4\hbar^2 r^4\alpha^2}{m} + \frac{1}{2}m\omega^2 r^2$$

ii) $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V u(r) = E\,u(r)$ $\qquad u(r) = r\cdot e^{-\alpha r^2}$

preciso achar $\boxed{E}$? ou só α e substituir E?

Q10. sistem contínuo

$P_2V - TS = -N_0 kT \ln Z$

$$Z = \int_V \ldots \iint_p \ldots \int_p e^{-\beta \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m}} d\vec{p}^{\,3}_{1 \ldots N} \, d\vec{r}^{\,3}_{1 \ldots N}$$

$$= \int_V \ldots \iint_p \ldots \int e^{-\beta \frac{\vec{p}_1^{\,2}}{2m}} \cdot e^{-\beta \frac{\vec{p}_2^{\,2}}{2m}} \cdots e^{-\beta \frac{\vec{p}_N^{\,2}}{2m}} d\vec{p}_1^{\,3} \ldots d\vec{p}_N^{\,3} \, d\vec{r}^{\,3}_{1 \ldots N}$$

Para cada partícula

$$\int_{-\infty}^{\infty} e^{-\beta \frac{p^2}{2m}} dp = \sqrt{\frac{\pi \, 2m}{\beta}}$$

$$\lambda = (2\pi m kT)^{\frac{1}{2}}$$

Para $p_x, p_y, p_z \Rightarrow Z = \left(\frac{2m\pi}{\beta}\right)^{\frac{3}{2}} \cdot \int \ldots \int d\vec{r}^{\,3}_{1 \ldots N} \Rightarrow \boxed{Z = \frac{V}{\lambda^3}}$

b) sistem N átomos

$$\boxed{Z = \frac{V^N}{\lambda^{3N}}} \Rightarrow Z = \frac{1}{N! \, h^{3N}} \left(\frac{V}{\lambda^3}\right)^N$$

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a) $H = (p_x^2 + p_y^2 + p_z^2)/2m$

$$Z = \int_V \int \iint e^{-\beta \sum_{i=1}^{3} p_i^2/2m} d\vec{p}_i^{\,3} \, d\vec{r}_i^{\,3} = \int \ldots \iint \ldots \int e^{-\beta \frac{p_x^2}{2m}} \cdot e^{-\beta \frac{p_y^2}{2m}} \cdot e^{-\beta \frac{p_z^2}{2m}} dp_x^3 \, dp_y^3 \, dp_z^3 \, dr_i^3$$

$$\int_{-\infty}^{\infty} e^{-\frac{\beta p^2}{2m}} dp = \sqrt{\frac{\pi \cdot 2m}{\beta}} \Rightarrow \boxed{Z = \frac{1}{h^3} \left(\frac{2m\pi}{\beta}\right)^{\frac{3}{2}} \cdot V}$$

S

b) $$Z = \frac{1}{N!}\left(\frac{2\pi m k_B T}{h^2}\right)^{3N/2} \cdot V^{3N}.$$

$$F = -\frac{1}{\beta}\ln Z = -\frac{1}{\beta}\cdot\left[\frac{3N}{2}\cdot\ln\left(\frac{2\pi m k_B T}{h^2\beta}\right) + 3N\ln V - \ln N!\right]$$

$$\Rightarrow f = \frac{F}{N} = \frac{-3k_BT}{2}\ln\left(\frac{2\pi m k_B T}{h^2}\right) - \cancel{3} k_BT\ln V + \frac{k_BT}{\cancel{N}}\overset{\substack{0 \text{ l'hôpital} \\ N\to\infty}}{\ln(N!)}$$

c) $$U = -\frac{\partial}{\partial\beta}\ln Z = \boxed{\frac{3}{2}k_BT}$$

$$\frac{3}{2}\cdot\frac{\cancel{h^2\beta}}{\cancel{2\pi m}}\cdot\left(-\frac{\cancel{2\pi m}}{\cancel{h^2}\beta^2}\right) = \boxed{\frac{3}{2}k_BT}$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = +\frac{3N}{\beta V} \Rightarrow \boxed{\frac{\cancel{3}k_BT}{v}}$$

d) $$S = -\left(\frac{\partial F}{\partial T}\right)_V = k_B\frac{3}{2}\ln T + k_B\ln v + \frac{3}{2}k_B + k_B C$$